Cyclic clique minors

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Sumary

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- a k-linked subgraph [Thomas and Wollan, 2005].

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- A large clique as a cyclic minor.

What is a lollipop?

Definition

A *lollipop* $L = (P, C)$ in a graph G is a pair where $P = p_1 \ldots p_s$ $(s \geq 1)$ is a path of G, $C = c_1 \ldots c_t c_1$ $(t \geq 3)$ is a cycle of G, $p_s = c_1$ and $V(P) \cap V(C) = \{c_1\}$;

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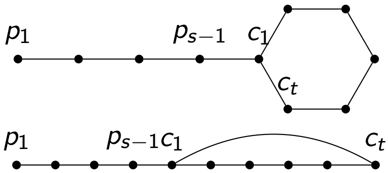

Figure: Two representations of a lollipop L.

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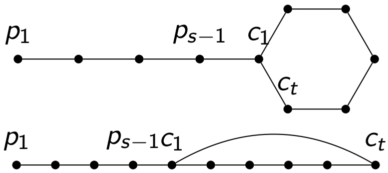

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Definition

A lollipop L is *optimal* if:

- L is vertex-wise maximal and
- among all lollipops on $V(L)$, L has a cycle of maximum length.

Hamiltonian paths

From now on we consider an optimal lollipop in a graph with minimum degree at least k.

Lemma

If $G[C]$ contains a Hamiltonian path with ends c_1 and u, then $N_G(u) \subseteq V(C)$.

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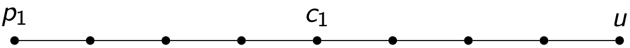

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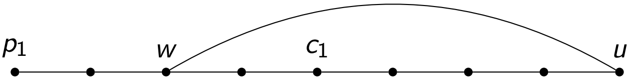

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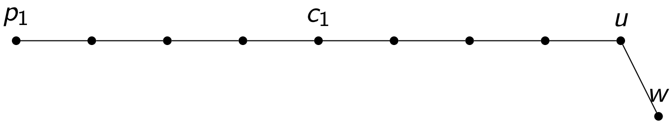

Active paths

Definition

- $\mathcal{S}_1 = \{c_1 c_2 \dots c_t, c_1 c_t c_{t-1} \dots c_2\}$.
- For all $i \geq 1$, define $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$.

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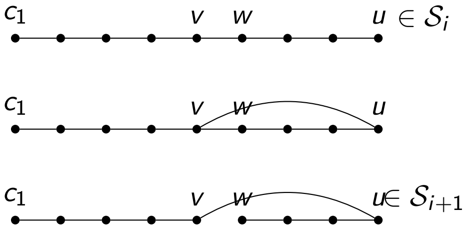

Figure: Construction of an element of $\mathcal{S}_{i+1}$ from an element of $\mathcal{S}_i$.

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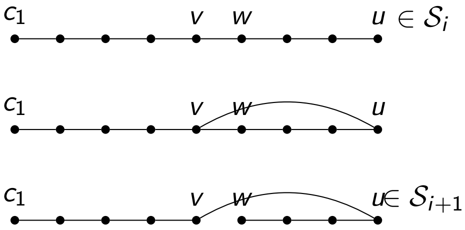
- An *active path* is any path Q from some $\mathcal{S}_i$;



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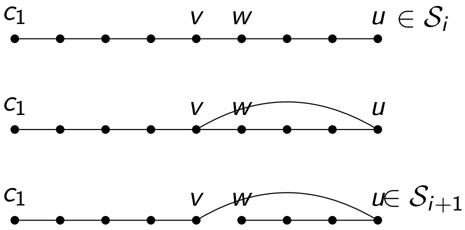

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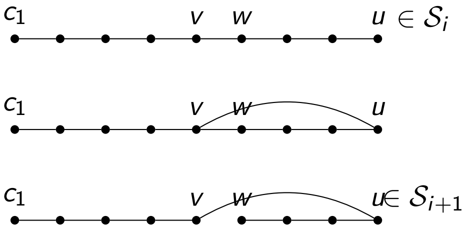

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An edge of C whose both ends are non-active is *passive*.

Active vertices

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C contains at least k active vertices. Moreover, if $d_C(c_1) < k$, then C contains at least $k+1$ active vertices.

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Theorem [Gupta, Kahn and Robertson, 1980]

C contains at least $k+1$ vertices of degree at least k (in C). In particular, C has at least $\frac{(k+1)(k-2)}{2}$ chords.

Lemma

If R is a subpath of C containing only passive edges and u is an active vertex, then u has at most one neighbor in R. Moreover, such a neighbor is an end of R.

Theorem

There exist sets $X_1, X_2 \subseteq E(C)$ so that for $i \in \{1, 2\}$, the graph G_i obtained by deleting all vertices not in C and contracting the edges of X_i has at least k vertices and

- minimum degree at least $\lceil \frac{k+2}{2} \rceil$ if $i = 1$, and
- average degree at least $\frac{2}{3}(k+1)$ if $i = 2$.

K_4 and K_5 as a cyclic minor

Lemma

Let C be a Hamiltonian cycle of a graph F. If F has minimum degree at least 3, then we can contract edges of C to obtain K_4. If F has average degree at least 6, then we can contract edges of C to obtain K_5.

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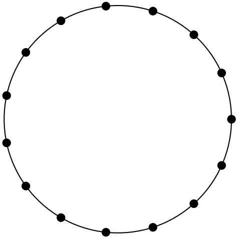

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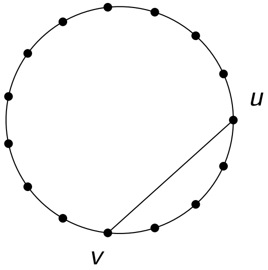

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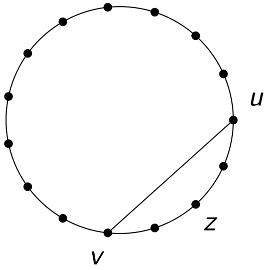

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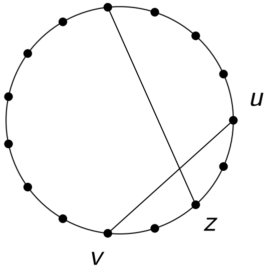

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Definition

Let $z_1, \ldots, z_t$ with $t \geq 3$ be the common neighbors of the ends of e in F, in order along the path $C - e$. We say that the vertices $z_2, \ldots, z_{t-1}$ are the *peaks* for e.

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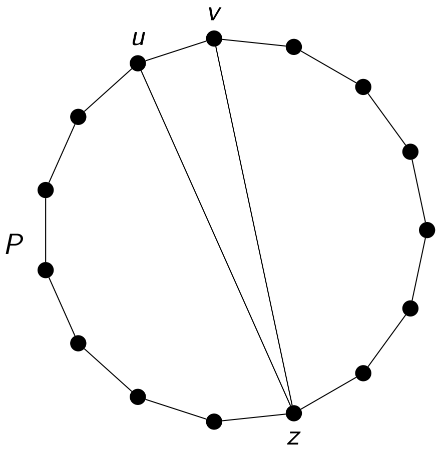

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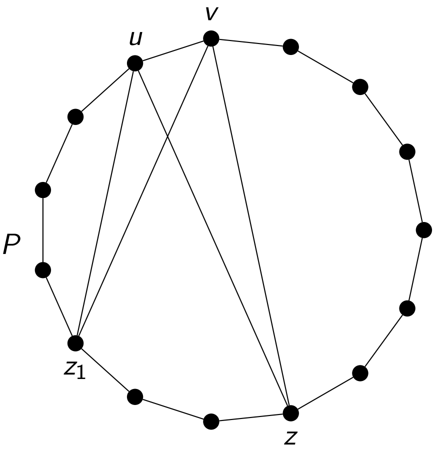

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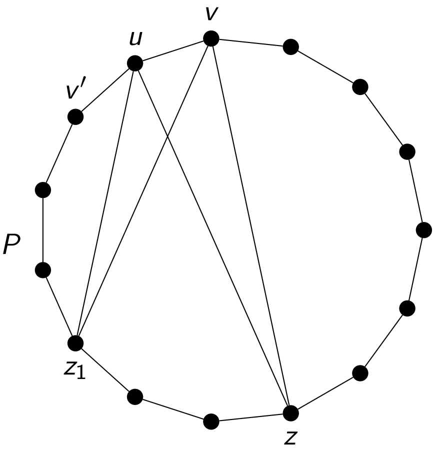

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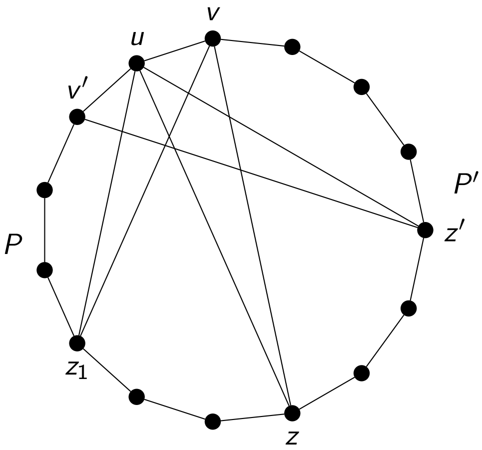

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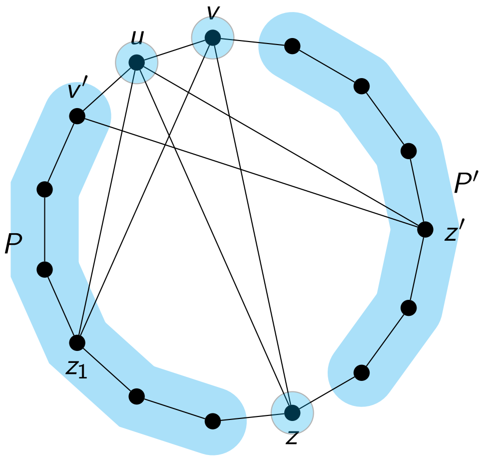

Figure: Model of a K_5 cyclic minor.

Upper bound for the minimum degree

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$f(k)$ is the smallest integer δ such that every graph of minimum degree at least δ contains K_k as a cyclic minor.

Corollary

$f(4) = 3$ and $f(5) \leq 8$.

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Lemma [Thomas and Wollan, 05]

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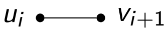

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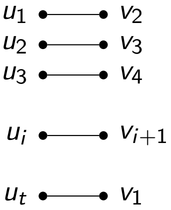

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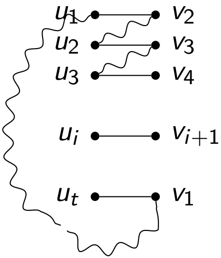

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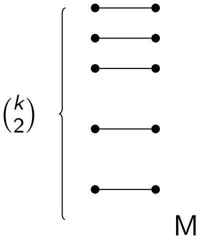

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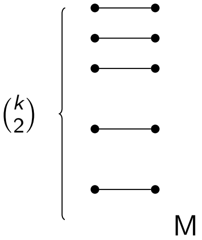

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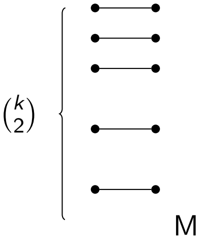
- $V_i = \{v_{(k-1)(i-1)+1}, \dots, v_{(k-1)(i-1)+(k-1)}\}$



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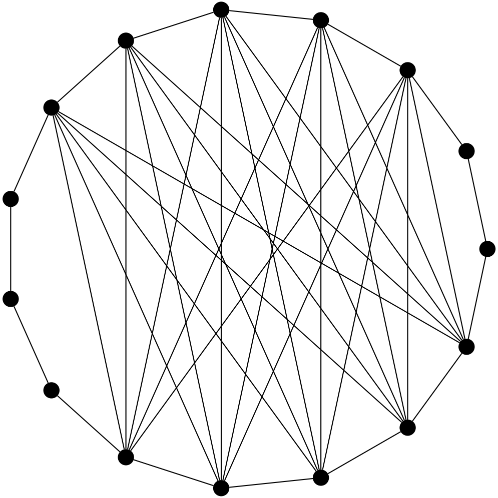

Figure: Cycle with chords in bipartite configuration.

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Is there a function f that tends to $+\infty$ such that every graph with minimum degree 3 contains a cycle of length ℓ with at least $f(\ell)$ chords?

Thank you!